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Representations of the Poincaré Semigroup and Relativistic Causality

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This paper provides the mathematical tools for addressing issues of two kinds of causality in relativistic scattering theory: general causality, i.e., an effect can only be measured after its cause, and Einstein causality, i.e., no propagation of probability outside of the forward light cone. Starting from Wigner's unitary irreducible representations of the Poincaré group for noninteracting, one particle states, we describe the mathematical tools necessary to describe scattering states, the Lippmann–Schwinger Dirace kets, and to describe resonances and decaying states, the relativistic Gamow ket. An important step for their derivations is the Hardy space hypothesis. Investigating the transformation properties of scattering and resonance states under the dynamical Poincaré semigroup reveals that both kinds of causality result from this hypothesis about nature of the spaces of states and observables.

KEY WORDS: relativistic causality; Poincaré semigroup; Gamow ket.

1. INTRODUCTION

Relativistic quantum fields are constructed from the unitary irreducible representations (UIR) of the Poincaré group $\mathcal{P}$ (Weinberg, 1995; Wigner, 1939, 1964) of space-time transformations. Elementary relativistic quantum systems are associated with the UIR characterized by real invariants m and j, representing the particle mass and spin.

The UIR of the Poincaré group describe stable elementary particles (stationary systems), but stable particles are in the minority. For longlived unstable particles, characterized by a small inverse lifetime Γ, the UIR can provide an approximate description. However, the process of decay cannot be modeled by the UIR. In the case of short-lived particles with large width Γ, such as resonances in a scattering experiment, the approximation of the particle by UIR becomes even more suspect. As a result, some have claimed that resonances are more complicated objects than stable particles and preclude their description by state vectors. Our opinion is the opposite: stable elementary particles are not qualitatively different from unstable

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particles, only quantitatively different in their value of Γ, with $\Gamma = 0$ for stable particles.

Previous work (Bohm *et al.*, 2000) provides a partial mathematical solution to this problem of representing relativistic unstable particles by introducing a special class of (nonunitary) semigroup representations of the Poincaré group. In (Bohm *et al.*, 2000), these representations were used to define relativistic Gamow vectors, allowing a description for unstable particles in terms of nonunitary irreducible representations (the semigroup representations) analogous to the description of stable particles by UIR. One application of these representations and the relativistic Gamow vector has been to the controversy over the parameterization of the lineshape of relativistic resonances.

After providing the necessary background, we wish to show how the transformation properties of the Poincaré *semigroup* representations allow for a causal theory of relativistic scattering and unstable particles. Issues of causality have vexed quantum theory for a long time, and the resolution of these issues is one of the strengths of using semigroup representations for the construction of the wave functions of unstable particles. What follows addresses both the general issue of causality and Einstein causality, i.e., no propagation faster than the speed of light, by looking at the Born probabilities for detecting decay products of unstable particles. A more detailed analysis can be found in (Bohm *et al.*, 2002); an analysis focusing more specifically on the causality issues is in preparation (Bohm and Harshman, in preparation).

2. TRANSFORMATION PROPERTIES OF THE ONE PARTICLE, NONINTERACTING REPRESENTATIONS OF THE POINCARÉ GROUP

Wigner's classic paper (Wigner, 1939, 1964) established that the interaction-free one particle states furnish a unitary irreducible representation (UIR) of the the Poincaré group $\mathcal{P}$. Here we will discuss only the projective representations of the semidirect product of the group of proper, orthochronous Lorentz transformations $\Lambda \in \mathrm{SO}(1,3)$ and the group of space–time translations $x \in \mathbb{R}^4$. Wigner classified the irreducible representations of $\mathcal{P}$ which are labeled by two number identified with the mass squared $\mathbf{s} = m^2 = p^\mu p_\mu$ and spin j of the particles.

The Dirac kets form a basis for a particular UIR to expand a given state vector ϕ of the UIR $(\mathbf{s}, j)$:

$$\phi = \sum_\xi \int \frac{d^3\hat{\mathbf{p}}}{2\hat{p}^0} |\hat{\mathbf{p}}, \xi(\mathbf{s}, j)\rangle\langle\hat{\mathbf{p}}, \xi(\mathbf{s}, j)|\phi\rangle, \tag{1}$$

where we have chosen the invariant measure

$$d\mu(\hat{\mathbf{p}}) = \frac{d^3\hat{\mathbf{p}}}{2\hat{p}^0}, \qquad \hat{p}^0 = \sqrt{1 + \hat{\mathbf{p}}^2}, \tag{2}$$

which implies the normalization

$$\begin{aligned}\langle \hat{\mathbf{p}}, \xi(\mathbf{s}, j) | \hat{\mathbf{p}}', \xi'(\mathbf{s}, j)\rangle &= 2\hat{p}^0 \delta^3(\hat{\mathbf{p}} - \hat{\mathbf{p}}')\delta_{\xi\xi'} \\ &= 2p^0 \mathbf{s} \delta^3(\mathbf{p} - \mathbf{p}')\delta_{\xi\xi'}. \end{aligned} \tag{3}$$

Here we use the 4-velocity basis of Dirac kets $|\hat{\mathbf{p}}, \xi(\mathbf{s}, j)\rangle$, where $\hat{p}_\mu = p_\mu/\sqrt{\mathbf{s}}$ and ξ is the discrete spin component along a particular axis, $-j < \xi < j$. Since the momentum is constrained to the mass shell in the UIR $(\mathbf{s}, j)$, the vectors can be labeled by the three spatial components $\hat{\mathbf{p}} = \mathbf{p}/m = \gamma\mathbf{v}$ of the 4-velocity $\hat{p} = p/m$, where $\gamma = 1/\sqrt{1-\mathbf{v}^2} = \hat{p}^0$. The 4-velocity basis is equivalent to the standard Wigner (momentum) basis $|\mathbf{p}, \xi(\mathbf{s}, j)\rangle$ except for normalization, however it will be preferable for constructing the minimally complex representations of the Poincaré semigroup (see below and reference (Bohm *et al.*, 2000) for further explanation and justification of the use of the 4-velocity basis).

The Dirac ket $|\hat{\mathbf{p}}, \xi(\mathbf{s}, j)\rangle$ is what we conceptually associate to a microphysical state, such as a particle which leaves a track in a detector, and are the units out of which quantum fields are built (Weinberg, 1995). However, the Dirac ket, and also the Dirac basis vector expansion (1), requires mathematical apparatus beyond the Hilbert space. The Dirac basis vector expansion does not hold for all vectors in the Hilbert space $\mathcal{H}(\mathbf{s}, j)$ of the Poincaré group, but only for a suitably chosen (i.e., nuclear) dense subset $\Phi \subset \mathcal{H}(\mathbf{s}, j)$. Then the linear topological dual of Φ, the space $\Phi^\times$ of antilinear functionals on Φ, contains the Dirac kets (Bohm and Gadella, 1989), and in particular the basis vectors for the complete set of commuting operators (CSCO) of the Poincaré algebra chosen for the expansion. Usually one takes for the space Φ the space of differentiable vectors of the unitary representation $U(\Lambda, x)$ and endows it with a countably-normed topology defined by the Nelson operator (Bohm, 1973); it is sufficient to specify the space Φ so that if $\psi \in \Phi \subset \mathcal{H}(\mathbf{s}, j)$, the functions $\psi(p)$ of the momentum p (or 4-velocity $\hat{p} = p/m$) are smooth rapidly decreasing functions (Schwartz space functions). Then the basis vectors $|\hat{\mathbf{p}}, \xi(\mathbf{s}, j)\rangle$ are elements of the dual space $\Phi^\times$ of the subset of differential vectors $\Phi \subset \mathcal{H}(\mathbf{s}, j)$. For vectors $\phi \in \Phi \subset \mathcal{H}(\mathbf{s}, j)$, Dirac's basis vector expansion holds.

Within this mathematical framework, the action of the Poincaré transformation $(\Lambda, x) \in \mathcal{P}$ on the Dirac kets can be mathematically defined and is given by the standard formula (Weinberg, 1995):

$$\begin{aligned} U^\times(\Lambda, x)|\hat{\mathbf{p}}, \xi(\mathbf{s}, j)\rangle &= e^{-ip\cdot x} \sum_{\xi'} D^j_{\xi'\xi}(W(\Lambda^{-1}, p))|\Lambda^{-1}\hat{\mathbf{p}}, \xi'(\mathbf{s}, j)\rangle \\ &= e^{-i\gamma\sqrt{\mathbf{s}}(t - \mathbf{v}\cdot\mathbf{x})} \sum_{\xi'} D^j_{\xi'\xi}(W(\Lambda^{-1}, p))|\Lambda^{-1}\hat{\mathbf{p}}, \xi'(\mathbf{s}, j)\rangle \end{aligned} \tag{4a}$$

$$\text{for} \quad \Lambda \in SO(1,3),\ t \in (-\infty, \infty), \quad \text{and} \quad x \in \mathbb{R}^3. \tag{4b}$$

Here $\mathbf{v}$ is the three velocity $\hat{\mathbf{p}} = \gamma\mathbf{v}$ and $W(\Lambda^{-1}, p) = L^{-1}(\Lambda\hat{p})\Lambda L(\hat{p})$ is the Wigner rotation. For later results, it is important to note that the standard boost $L(\hat{p})$ and therewith $W(\Lambda^{-1}, p)$ depends upon $\hat{p}$ not upon the momentum $p = \sqrt{\mathbf{s}}\hat{p}$.

Usually in place of $U^{\times}(\Lambda, x)$ one writes $U^{\dagger}(\Lambda, x) = U^{-1}(\Lambda, x) = U^{-1}(\Lambda^{-1}, \Lambda^{-1}x)$. However, since in (4) the basis vectors are Dirac kets, $|\hat{\mathbf{p}}, \xi(\mathbf{s}, j)\rangle \in \Phi^{\times} \supset \mathcal{H}(\mathbf{s}, j)$, we use the notation $U^{\times}(\Lambda, x)$. The superscript $\times$ denotes the unique extension of the unitary operator $U^{\dagger}(\Lambda, x) \subset U^{\times}(\Lambda, x)$ to the dual space of functionals $\Phi^{\times}$ of a Gelfand triplet:

$$\Phi \subset \mathcal{H}(\mathbf{s}, j) \subset \Phi^{\times}. \tag{5}$$

3. THE LIPPMANN–SCHWINGER DIRAC KETS AND RELATIVISTIC GAMOW KETS

For our analysis of unstable states and causality, we need the transformation properties for the interacting, multiparticle states, not just for the one particle, interaction-free kets. The Lippmann–Schwinger kets are the *interacting* in- and out-plane wave states of, for example, a scattering experiment with formation of a resonance R by the process:

$$1 + 2 \to R \to 3 + 4. \tag{6}$$

The Lippmann–Schwinger Dirac kets are used in the expansion of the in-states $\phi = \phi_1 \times \phi_2$ and out-states (out-observables) $\psi = \psi_3 \times \psi_4$, such as in the S-matrix scattering amplitude

$$\begin{aligned}(\psi^{\text{out}}, S\phi^{\text{in}}) &= (\psi^{-}, \phi^{+}) \\ &= \sum_{j,\xi} \int_{s_{\min}}^{\infty} ds \int d\mu(\hat{\mathbf{p}}) \langle^{-}\psi|\hat{\mathbf{p}}, \xi(\mathbf{s}, j)^{-}\rangle S_j(\mathbf{s}) \langle^{+}\hat{\mathbf{p}}, \xi(\mathbf{s}, j)|\phi^{+}\rangle.\end{aligned} \tag{7}$$

To construct the incoming and outgoing Lippmann–Schwinger kets for processes like (6) (and eventually to construct the relativistic Gamow vector), we start with the the direct product spaces $\mathcal{H}_{12} \equiv \mathcal{H}(m_1^2, j_1) \oplus \mathcal{H}(m_2^2, j_2)$ and $\mathcal{H}_{34} \equiv \mathcal{H}(m_3^2, j_3) \oplus \mathcal{H}(m_4^2, j_4)$ of the two incoming and outgoing particles.[2] The direct product of the representation spaces for particles 1 and 2 is not irreducible, but can be broken into a direct sum of UIRs:

$$\begin{aligned}\mathcal{H}_{12} &\equiv \mathcal{H}\big(m_1^2, j_1\big) \oplus \mathcal{H}\big(m_2^2, j_2\big) \\ &= \int_{(m_1+m_2)^2}^{+\infty} ds \bigoplus_{j=0}^{\infty} \bigoplus_{\{n\}} \mathcal{H}^n(\mathbf{s}, j),\end{aligned} \tag{8}$$

where $\mathbf{s} = p^{\mu}p_{\mu} = (p_1 + p_2)^{\mu}(p_1 + p_2)_{\mu}$ is the total invariant mass square of the system of particles, $\mathbf{s}_{\min} = (m_1 + m_2)^2$, j is the total spin and the set n carries the

[2] Processes involving more than two body reactions can be handled similarly, but the Clebsch–Gordan coefficients for combining more than two representations quickly become unwieldy.

information of the masses and spins $\{m_1, m_2, j_1, j_2\}$ of the original particles and also the degeneracy labels arising from the angular momentum coupling scheme (such as spin–orbit or helicity coupling).

As with the Dirac kets for one particle, interaction-free kets, Dirac kets for the two-particle (interaction-free) are not vectors in the Hilbert space $\mathcal{H}_{12}$. Defining again a dense subspace of $\mathcal{H}_{12}$ of vectors whose realizations are sufficiently well-behaved functions of the spatial components of the 4-velocity and also the inverse mass-squared $\mathbf{s}$, another Gelfand triplet for the direct product space can be defined and the velocity eigenkets $|\hat{\mathbf{p}}, \xi(\mathbf{s}, j, n)\rangle$ provide a basis for the dense subspace of each UIR in the direct sum. They are eigenkets of the total energy-momentum operators $(H = P^0, \mathbf{P})$ with eigenvalues

$$p^\mu = (p_1 + p_2)^\mu = (p_3 + p_4)^\mu, \mathbf{s} = p_\mu p^\mu, (\sqrt{\mathbf{s}_1} + \sqrt{\mathbf{s}_2})^2 \leq \mathbf{s} < \infty. \quad (9)$$

Clebsch–Gordan coefficients for the Poincaré group connect the direct product basis kets $|\hat{\mathbf{p}}_1, \xi_1(\mathbf{s}_1, j_1); \hat{\mathbf{p}}_2, \xi_2(\mathbf{s}_2, j_2)\rangle$ to the velocity eigenkets of the irreducible representations of the direct product space $|\hat{\mathbf{p}}, \xi(\mathbf{s}, j, n)\rangle$. (For the velocity basis Clebsch–Gordan coefficients, see (Bohm and Kaldass, 1999); see the references of that article for the classic papers.)

To incorporate the effects of the interaction (and the boundary conditions of a scattering experiment), the Lippmann–Schwinger Dirac ket basis can be constructed from the eigenkets of the UIR of the direct product spaces. The in- and out-Lippmann–Schwinger scattering states are defined in formal scattering theory by applying the Møller operators $\Omega^\pm$ to the rest velocity eigenkets ($\hat{\mathbf{p}} = \mathbf{0}$),

$$\begin{aligned} |\mathbf{0}, \xi(\mathbf{s}, j)^\pm\rangle &= \Omega^\pm|\mathbf{0}, \xi(\mathbf{s}, j)\rangle \\ &= \left(1 + \frac{1}{\sqrt{\mathbf{s}} - H \pm i\epsilon}\right)|0, \xi(\mathbf{s}, j)\rangle \end{aligned} \quad (10)$$

where H is the exact interaction Hamiltonian. They provide an alternate basis for the UIR of the two-particle system labeled by $(\mathbf{s}, j)$ and are eigenkets of the interaction–incorporating operators $\hat{\mathbf{p}} = \mathbf{p}/\sqrt{\mathbf{s}}$. We will assume that the total interacting system remains Poincaré invariant and so the total operators of the interacting algebra are the exact generators of the Poincaré transformations for the interacting (two-particle) system (Weinberg, 1995). Because of this, the Lippmann–Schwinger kets $|\hat{\mathbf{p}}, \xi(\mathbf{s}, j)^\pm\rangle$ with any velocity $\hat{\mathbf{p}}$ can be obtained from $|\mathbf{0}, \xi(\mathbf{s}, j)^\pm\rangle$ by applying a Lorentz transformation by the standard boost $U(L(\hat{p}))$ whose parameters are the 4-velocities $\hat{p}$. This assumption, however, is not universally accepted and may be an oversimplification, especially in the case where the interaction can couple a state to infinitely many zero-mass representations, i.e., the "infraparticle problem" (Haag, 1996; Schroer, 1963).

From the Lippmann–Schwinger Dirac kets, it becomes possible to construct the relativistic Gamow vector to represent resonances and decaying states (Bohm *et al.*, 2000, 2002). If there is a resonance R of spin j in the scattering process

(6), then the j-th partial S matrix $S_j(\mathbf{s})$ of (7) has a pole in the second sheet of the lower half plane at $\mathbf{s} = \mathbf{s}_R = (M - i\Gamma/2)^2$. We choose to parameterize the complex pole position $\mathbf{s}_R$ by two real parameters (M, Γ) in this way to ensure the consistency of the relation $\Gamma = \hbar/\tau$, where τ is the lifetime of the resonance (Bohm and Harshman, 2000). By contour integration around the pole (which requires analyticity conditions for the spaces of states $\{\phi^+\} \in \Phi_-$ and observables $\{\psi^-\} = \Phi_+$, described below), one defines a generalized state vector (ket) which can be shown (Bohm *et al.*, 2002) to have a relativistic Breit–Wigner energy distribution

$$\begin{aligned}\langle\psi^-|\mathbf{p}, \xi(\mathbf{s}_R, j)\rangle &= -\frac{i}{2\pi}\oint d\mathbf{s}\langle\psi^-|\hat{\mathbf{p}}, \xi(\mathbf{s}, j)^-\rangle\frac{1}{\mathbf{s}-\mathbf{s}_R}\\ &= \frac{i}{2\pi}\int_{-\infty II}^{+\infty II} d\mathbf{s}\langle\psi^-|\hat{\mathbf{p}}, \xi(\mathbf{s}, j)^-\rangle\frac{1}{\mathbf{s}-\mathbf{s}_R} \quad \text{for all} \quad \psi^- \in \Phi_+,\end{aligned} \tag{11}$$

and which corresponds to a relativistic Breit–Wigner amplitude in the j-th partial scattering amplitude. This ket $|\hat{\mathbf{p}}, \xi(\mathbf{s}_R, j)^-\rangle$ which we call a relativistic Gamow vector, is a generalized eigenvector of the self-adjoint total invariant mass-squared operator $P_\mu P^\mu = H^2 - \mathbf{P}^2$ with complex eigenvalue $\mathbf{s}_R = (M - i\Gamma/2)^2$. It fulfills the same conceptual role as the Dirac ket did for stable particles: we identify it with the microphysical state of a decaying particle or resonance.

4. TRANSFORMATION PROPERTIES OF THE LIPPMANN–SCHWINGER DIRAC KETS AND THE RELATIVISTIC GAMOW KET

In the previous section, we have applied the Møller operators to the Dirac eigenkets to produce the Lippmann–Schwinger scattering states, and from them defined the relativistic Gamow vector for representing resonances and decaying states. We did this without considering what analytical properties are required of the wave functions $\langle^-\hat{\mathbf{p}}, \xi(\mathbf{s}, j)|\psi^-\rangle$ and $\langle^+\hat{\mathbf{p}}, \xi(\mathbf{s}, j)|\phi^+\rangle$ in (7) for these manipulations to make sense. We can think of the restrictions required for the spaces of states and observables as the boundary conditions imposed by the physical nature of the scattering experiment. The important result of this paper is that the boundary conditions required by mathematical rigor place constraints on the allowable Poincaré transformations and that these constraints exactly coincide with the two notions of causality described in the introduction.

The boundary conditons for the in- and out-Lippmann–Schwinger kets are expressed by the infinitesimal $\pm i\epsilon$ in the Møller operators (10), which we take as the imaginary part of the invariant energy[3] $\sqrt{\mathbf{s}} \to \sqrt{\mathbf{s}} \pm i\epsilon$. This requires analytic

[3] The usual choice is to give the infinitesimal imaginary part to the component $p^{\mathbf{0}}$ of the 4-momentum, but for infinitesimal ϵ it does no matter and we shall therefore give the infinitesimal imaginary part to the invariant mass $\sqrt{\mathbf{s}}$ because that can be more easily generalized to finite $\epsilon \to \Gamma/2$.

continuability of the wave functions $\langle^+\hat{\mathbf{p}}, \xi(\mathbf{s}, j)|\phi^+\rangle$ and of the complex conjugate wave functions $\langle^-\psi\hat{\mathbf{p}}, \xi(\mathbf{s}, j)^-\rangle = \overline{\langle^-\hat{\mathbf{p}}, \xi(m^2, j)|\psi^-\rangle}$ at least into the top strip of the lower complex energy semiplane. This requirement on the wave functions is stronger than that required for the definition of the Dirac kets, which required that $\langle\hat{\mathbf{p}}, \xi(\mathbf{s}, j)|\phi\rangle$ are smooth, rapidly-decreasing functions of $\mathbf{s}$ (and $\hat{\mathbf{p}}$) (Schwartz space functions) and not necessarily analytic. Instead, the scattering in- and out-wave functions $\langle^+\hat{\mathbf{p}}, \xi(\mathbf{s}, j)|\phi^+\rangle$ and $\langle^-\hat{\mathbf{p}}, \xi(\mathbf{s}, j)|\psi^-\rangle$ must be restricted to smaller subspaces which have sufficient analyticity properties from below or above. These we call Φ_- and Φ_+, respectively. Then, while the Dirac kets in (4) are elements of $\Phi^\times$, the Lippmann–Schwinger kets are the elements of the duals of these spaces, i.e., $|\hat{\mathbf{p}}, \xi(\mathbf{s}, j)^\mp\rangle \in \Phi_\pm^\times$.

As a hypothesis, we take for $\Phi_\pm$ the Hardy spaces of the upper and lower complex $\mathbf{s}$-semiplane, i.e., spaces whose wave functions are analytic in the upper (Φ_+) or lower (Φ_-) complex $\mathbf{s}$-semiplanes. This is more than one needs for the Lippmann–Schwinger kets with infinitesimal Im$\mathbf{s} = \mp i\epsilon$, however to apply (10) one definitely needs *some* analyticity property of the wave functions $\langle^-(\mathbf{s}, j)\mathbf{p}, \xi|\psi^-\rangle$ in the upper and $\langle^+(\mathbf{s}, j)\mathbf{p}, \xi|\psi^+\rangle$ in lower semiplane and cannot addmit all functions of the Schwartz space as wave functions. In addition, the choice of Hardy space functions will allow us also to continue the S matrix in (7) into the *lower* $\mathbf{s}$-semiplane, e.g., to the pole position $\mathbf{s}_R = (M_R - i\Gamma/2)^2$ of the resonance R in (6) and define the relativistic Gamow vector.

Therefore we replace the usual axiom of Hilbert space quantum mechanics

$$\{\text{set of in-states}\,\phi^+\} = \{\text{set of out states}\,\psi^-\} = \mathcal{H}, \tag{12a}$$

or, when Dirac kets are incorporated,

$$\{\text{set of in-states}\,\phi^+\} = \{\text{set of out states}\,\psi^-\} = \Phi \subset \mathcal{H} \subset \Phi^\times \tag{12b}$$

by the new hypothesis

$$\{\text{set of prepared in-states}\,\phi^+\} \equiv \Phi_- \subset \mathcal{H} \subset \Phi_-^\times \ni |\hat{\mathbf{p}}, \xi(\mathbf{s}, j)^+\rangle \tag{13a}$$

$$\{\text{set of detected out-states}\,\psi^-\} \equiv \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times \ni |\hat{\mathbf{p}}, \xi(\mathbf{s}, j)^-\rangle \tag{13b}$$

where Φ_- and Φ_+ are two different dense subspaces of the same Hilbert space $\mathcal{H}$.

Since the spaces Φ_+ (Φ_-) are smaller than Φ, the space $\Phi_+^\times$ ($\Phi_-^\times$) are larger than the space $\Phi^\times$ (tempered distributions) and the $|\hat{\mathbf{p}}, \xi(\mathbf{s}, j)^-\rangle \in \Phi_+^\times$ are "more generalized" than the Dirac kets. As a consequence, the transformation formula (4a) that holds for (4b) does not hold any more for extensions of the unitary operator $U^\dagger(\Lambda, x)$ in $\mathcal{H}$ to the operators $U_\pm^\times(\Lambda, x)$ in $\Phi_\pm^\times$ of (13).

From the analyticity conditions (consider the factor $\exp(-ip \cdot x)$), it can be shown that the extension $U^\dagger(\Lambda, x) \subset U_+^\times(\Lambda, x)$ is defined only for transformations into the forward light cone (Bohm *et al.*, 2002). This means the transformations $U_+^\times(\Lambda, x)$ in $\Phi_+^\times$ (and therewith also the transformations $U_+(\Lambda, x) = U(\Lambda, x)|_{\Phi_+}$ in the space Φ_+ of out-states ψ^-) do not furnish a representation of the Poincaré group $\mathcal{P}$ but only a representation $(\mathbf{s}, j)$ of the Poincaré *semigroup* $\mathcal{P}_+$ of proper othrochronous Lorentz transformations and space–time translations into the forward light cone:

$$\mathcal{P}_+ = \left\{(\Lambda, x)|\Lambda \in \overline{\mathrm{SO}(3,1)}, \det\Lambda = +1, \Lambda^0_0 \geq 1; x^2 \equiv t^2 - \mathbf{x}^2 \geq 0, t \geq 0\right\}. \tag{14}$$

For the transformation property of the Lippmann–Schwinger ket $|\hat{\mathbf{p}}, \xi(\mathbf{s}, j)^-\rangle \in \Phi_+^\times$ under Poincaré transformations one can prove (Bohm *et al.*, 2002) the analogue of (4a):

$$\begin{aligned}\langle U_+(\Lambda, x)\psi^- || \mathbf{p}, \xi(\mathbf{s}, j)^-\rangle &\equiv \langle \psi^- | U_+^\times(\Lambda, x)|\mathbf{p}, \xi(\mathbf{s}, j)^-\rangle \\ &= e^{-ip\cdot x}\sum_{\xi'} D^j_{\xi'\xi}(W(\Lambda^{-1}, p))\langle\psi^-|\Lambda^{-1}\hat{\mathbf{p}}, \xi'(\mathbf{s}, j)^-\rangle\end{aligned} \tag{15}$$

for all $\psi^- \in \Phi_+$, but *only* for $t \geq 0$ and $t^2 - \mathbf{x}^2 \geq 0$. An analogous statement holds for $|\mathbf{p}, \xi(\mathbf{s}, j)^+\rangle \in \Phi_-^\times$ and the transformation $U_-^\times(\Lambda, x)$ in $\Phi_-^\times$ defined for $(\Lambda, x) \in \mathcal{P}_-$ (Poincaré transformations into the backward light cone).

For relativistic Gamow vectors $|\hat{\mathbf{p}}, \xi(\mathbf{s}_R, j)^-\rangle \in \Phi_+^\times$, defined by (11) from the Lippmann–Schwinger Dirac kets $|\hat{\mathbf{p}}, \xi(\mathbf{s}, j)^-\rangle \in \Phi_+^\times$, the transformation of $|\mathbf{p}, \xi(\mathbf{s}, j)^-\rangle$ under $(\Lambda, x) \in \mathcal{P}_+$ has similar form and restrictions:

$$\begin{aligned}&U_+^\times(\Lambda, x)|\mathbf{p}, \xi(\mathbf{s}_R, j)^-\rangle \\ &\quad = e^{-i\gamma\sqrt{\mathbf{s}_R}(t - \mathbf{x}\cdot\mathbf{v})}\sum_{\xi'} D^j_{\xi'\xi}(W(\Lambda^{-1}, p))|\Lambda^{-1}\hat{\mathbf{p}}, \xi'(\mathbf{s}_R, j)^-\rangle \\ &\text{only for} \quad t \geq 0 \quad \text{and for} \quad t^2 - \mathbf{x}^2 \geq 0,\end{aligned} \tag{16}$$

which is again understood as a functional equation over all $\psi^- \in \Phi_+$ as in (15). The property that $W(\Lambda^{-1}, p)$ depends upon $\hat{p}$ not upon the momentum $p = \sqrt{\mathbf{s}}\hat{p}$ is what allowed us to construct the "minimally complex" representations $(\mathbf{s}_R, j)$ by making the analytic continuations of the Lippmann–Schwinger kets to the Gamow kets

$$|b, \xi(\mathbf{s}, j)^-\rangle \to |b, \xi'(\mathbf{s}_R, j)^-\rangle \tag{17}$$

in such a way that the $b = \hat{p}$ remain unaffected, i.e., $\hat{p}$ is real. Therefore, the Gamow kets furnish an irreducible representation space characterized by $(\mathbf{s}_R, j)$ of the Poincaré semigroup $\mathcal{P}_+$ into the forward light cone.

The transformation property of the Gamow kets (16) is the same as that of the Lippmann Schwinger kets (15); the only difference is the limit $-i\Gamma/2 \to i0$. Superficially, (15,16) agree with Wigner's standard formula (4), except that they only hold for Poincaré transformations into the forward light cone and not for the whole Poincaré group. The mathematical derivation of (15) and (16) under (a precise version of) the hypothesis (13) is given in Bohm *et al.* (2002).

The Wigner representation (4) is for the symmetry transformation of an isolated stable particle, the semigroup representation in (15,16) is for the evolution of an interacting system, e.g., the evolution of a decaying particle due to some interaction. In the nonrelativistic case, the dynamics of this evolution is the time translation due to the total Hamiltonian; in the relativistic case the dynamics is the transformations into the forward light cone (14). The transformation (4) are just kinematic transformations and often one associates the Poincaré group representations ($\mathbf{s} = m^2,\ j$) just to interaction-free "asymptotic" states. The Poincaré transformations for the interacting system (15,16) with exact interaction–incorporating generators (9) describe the *dynamical* evolution. Experimental verification is possible only for invariance with respect to the semigroup transformations (14) because all physical transformations are necessarily into the forward light cone.

5. TRANSFORMATIONS AND CAUSALITY

The semigroup restriction, expressed as

$$t \geq 0 \tag{18a}$$

$$t^2 \geq \mathbf{x}^2 \equiv r^2/c^2, \tag{18b}$$

adds a new aspect to the dynamical evolution of the states and observables by Poincaré transformations (4); it results in the incorporation of causality into quantum mechanical probabilities.

The restrictions (18) are mathematical consequences of the Hardy space hypothesis (13). In turn, the hypothesis (13) is to a large extent suggested by the in- and out-going boundary conditions of the Lippmann–Schwinger kets. In addition to not fitting into the Hilbert space framework of quantum mechanics (12a) because they are kets, the Lippmann–Schwinger kets cannot be made compatible with an axiom like (12b) which allows only time-symmetric Dirac kets (e.g., Schwartz space functionals).

We now want to consider the consequences for calculating the probabilities of detecting the decay products of decaying states or resonances where the hypothesis (12) is replaced with (13) but where none of the other principles of quantum mechanics are modified. In particular, we shall retain the standard dynamical differential equations: the Schrödinger equation for states (prepared in-states) ϕ^+ and the Heisenberg equation for observables (detected out-states) ψ^-. Using the solutions of these equations $\phi^+(t)$ and $\psi^-(t)$, we can calculate the Born probabilities as a function of time, i.e., the probability to detect the observable $|\psi^-\rangle\langle\psi^-|$ in a state ϕ^+ at time t:

$$\mathcal{P}(t) = \mathcal{P}_{\phi^+}(|\psi^-(t)\rangle\langle\psi^-(t)|) = |\langle\psi^-(t)|\phi^+\rangle|^2 \quad \text{(Heisenberg picture)}$$
$$= \mathcal{P}_{\phi^+(t)}(|\psi^-\rangle\langle\psi^-|) = |\langle\psi^-|\phi^+(t)\rangle|^2 \quad \text{(Schrödinger picture).} \qquad (19)$$

Within the formalism of the time-symmetric Gelfand triplet, hypothesis (12b), the Born probability can be generalized to incorporate the Dirac kets, giving the interpretation of probability density to quantities such as $|\langle\phi|\hat{\mathbf{p}}\rangle|$ where $\phi \in \Phi$ and $|\hat{\mathbf{p}}\rangle \in \Phi^\times$.

At issues is the fact that the solutions to the dynamical equations $\phi^+(t)$ and $\psi^-(t)$ depend on the properties of the spaces to which ϕ^+ and ψ^- belong, the boundary conditions for the problem. The dynamical equations can be solved with initial condition $\phi^+, \psi^- \in \mathcal{H}$ from (12a) (or equivalently under assumption (12b)) and then the solutions are

$$\phi^+(t) = U^\dagger(t)\phi^+ = e^{-iHt}\phi^+ \quad \text{for} \quad -\infty < t < \infty$$
$$\psi^-(t) = U(t)\psi^- = e^{iHt}\psi^- \quad \text{for} \quad -\infty < t < \infty, \qquad (20)$$

where the different sign in the exponent derives from the different signs in the Heisenberg and Schrödinger equations. The result that the dynamical equations integrate to a unitary, reversible time evolution for Hilbert space vectors was proved by Stone and von Neumann (Stone, 1932; von Neumann, 1931). Using the solutions (20), the probabilities (19) can then be calculated for given $\phi^+, \psi^- \in \mathcal{H}$ for all time.

Alternatively, the Hardy space hypothesis (13) can provide the boundary conditions for the solution, giving the result

$$\phi^+(t) = e^{-iH_-t}\phi^+ \quad \text{for} \quad 0 \le t < \infty$$
$$\psi^-(t) = e^{-iH_+t}\psi^- \quad \text{for} \quad 0 \le t < \infty. \qquad (21)$$

To be precise, we have used for the generators of time translation the operators $H_\pm$, the restrictions of the self-adjoint Hilbert space operator H to the two (different, but each dense in $\mathcal{H}$) subspaces $\Phi_\pm$. These solutions $\phi^+(t)$ and $\psi^-(t)$ are the semigroup solutions of the the Schrödinger and Heisenberg equations and the restriction to positive times is the consequence of (13).[4] As a result, using the solutions (21), the probabilities (19) are predicted for $\phi^+ \in \Phi_-$ and $\psi^- \in \Phi_+$ *only* for $t \ge 0$, and not for all time as they were for the time-symmetric hypothesis (12).

[4] For example, according to (13) all out-states ψ^-, including the time evolved states $\psi^-(t) = \exp(iH_+t)\psi^-$ for any given $\psi^- \in \Phi_-$, must also be elements of the Hardy space Φ_+ and this is only fulfilled for $0 \le t < \infty$. (Consequently only for those times is the conjugate operator $(\exp(iH_+t))^\times = \exp(-iH_+^\times t)$ defined as a continuous operator on $\Phi_+^\times$.) A similar argument holds for $\exp(-iH_-t)\phi^+$ and $\phi^+ \in \Phi_-$. This result can be extended to general space–time translations giving the forward light cone restrictions (18).

This restriction to $t \geq 0$ of the probabilities has a clear interpretation if we extend the probability interpretation of (19) to include the relativistic Gamow ket $|\hat{\mathbf{p}}, \xi(\mathbf{s}_R, j)^-\rangle \in \Phi_+^\times$ representing a resonance or decaying state, in the same way as it can be generalized to include Dirac ket and probability densities. The quantity $\langle\psi^-|\hat{\mathbf{p}}, \xi(\mathbf{s}_R, j)^-\rangle$ is the probability amplitude to detect the observable (out-state) $|\psi^-\rangle\langle\psi^-|$ in the Gamow ket $|\hat{\mathbf{p}}, \xi(\mathbf{s}_R, j)^-\rangle \in \Phi_+^\times$ which represents the resonance R. We can calculate this amplitude for the Poincaré-transformed decay product $U(\Lambda, x)\psi^-$ in the generalized rest state $\psi^G = |\mathbf{0}, \xi(\mathbf{s}_R, j)^-\rangle$. Choosing for the Poincaré transformation the time translation $(\Lambda, x) = (1, (t, \mathbf{0}))$, we obtain from (16) the following result, valid only for $t \geq 0$:

$$\begin{aligned}\langle e^{iH_+t}\psi^-|\mathbf{0}, \xi(\mathbf{s}_R, j)^-\rangle &= \langle U_+(1, (t, \mathbf{0}))\psi^-|\mathbf{0}, \xi(\mathbf{s}_R, j)^-\rangle \\ &= \langle\psi^-|U_+^\times(1, (t, \mathbf{0}))|\mathbf{0}, \xi(\mathbf{s}_R, j)^-\rangle = \langle\psi^-|\psi^G(t)\rangle \\ &= e^{-i\gamma\sqrt{\mathbf{s}_R}t}\langle\psi^-|\mathbf{0}, \xi(\mathbf{s}_R, j)^-\rangle \\ &= e^{-iM_Rt}\, e^{-\Gamma t/2}\langle\psi^-|\mathbf{0}, \xi(\mathbf{s}_R, j)^-\rangle. \end{aligned} \tag{22}$$

For precision, $U_+(\Lambda, x)$ denotes the restriction of $U(\Lambda, x)$ to the dense subspace Φ_+ and $U_+^\times(\Lambda, x)$ denotes the conjugate of $U_+(\Lambda, x)$ in $\Phi_+^\times$ which is the uniquely defined extension of the unitary operator $U^\dagger(\Lambda, x)$, but only for x satisfying (18).

The result (22) implies that the probability rate to count ψ^- in a time interval Δt around the time t in the decay of a resonance described by ψ^G decreases exponentially in time:

$$|\langle\psi^-(t)|\psi^G\rangle|^2 = e^{-\Gamma t}|\langle\psi^-|\psi^G\rangle|^2 \quad \text{for} \quad t \geq 0, \tag{23}$$

but no probability is predicted for $t < 0$. What is this time $t = 0$ that the semigroup evolution (21) distinguishes? For states represented by the Gamow ket, this time can be identified with the finite time t_0 at which a state has been created and subsequently decays, for the K^0 in the process $\pi^- p \to \Lambda K^0$, $K^0 \to \pi^+\pi^-$ or the excited atom in $e^- A \to e^- A^*$, $A^* \to A\gamma$. Even though the production process takes finite time, this time can be very short compared to the time scale of decay process, as in the previous examples. In such cases, the mathematical semigroup time $t = 0$ is identfied with the decaying state production time t_0. This means an ensemble of single micro systems (e.g., K^0's or A^*'s) is created at an "ensemble" of times t_0 $(t_0^1, t_0^2, \ldots, t_0^N)$ connected to an ensemble of subsequent decay events at times in the rest frame of the decaying state $t(t^1, t^2, \ldots, t^n)$. All of these times t_0^i are mapped to the mathematical time $t = 0$ which appears in the semigroup time evolution (Bohm, 1999).

Given this interpretation, the fact that using the Hardy space hypothesis the theory provides no prediction for probabilities with $t < 0$ is a sensible result and this is our motivations for investigating the semigroup solutions (21) and

Poincaré transformation properties (15,16). However, the standard hypothesis of time-symmetric quantum mechanics (12), predicts time evolution of both the state and observable for any positive *and* negative time. Consequently, the probabilities for $\psi(t)$ in ϕ (19) are given for any time $-\infty < t < \infty$. Further for $\phi, \psi \in \mathcal{H}$, one can prove (Hegerfeldt, 1994, 1995, 1998) that

$$\text{either} \quad \mathcal{P}_\phi(|\psi(t)\rangle\langle\psi(t)|) \quad \text{is different from} \quad -\infty < t < \infty \tag{24a}$$

zero for (almost) all time

$$\text{or} \quad \mathcal{P}_\phi(|\psi(t)\rangle\langle\psi(t)|) \equiv 0 \quad \text{for all time} \quad -\infty < t < \infty. \tag{24b}$$

In the case (24b), there are no transitions and thus no decays, Therefore, if one wants transitions and chooses the solutions (20), then only the option (24a) remains, but this has problematic consequences:

(i) It predicts nonzero probabilities for precursor events, such as decay products detected at times $t < t_0$ before the state was prepared at time t_0.
(ii) It predicts nonzero probabilities for detecting decay products for times $t < t_0 + r/c$ for any finite but arbitrary distance r, or equivalently detection events at times and locations $r > c(t - t_0)$ outside the forward light cone, which violates Einstein causlity.

This latter consequence (ii) was the main concern of Hegerfeldt (1994, 1995, 1998) though the former consequence (i) more directly conflicts with the concept of causality in general since it does not predict that "something propagates faster than light" but that finite transition probability exists for $t < t_0$. This is an obviously unphysical prediction since a state needs to be prepared before an observable can be measured in it. Buchholz and Yngvason (1994) question whether such a simplistic construction for the transition probability is a physically realistic, useful theoretical construct; within the framework of time asymmetric RHS quantum theory (summarized by hypothesis (13) the transition amplitude has its common, intuitive meaning.

The problem (i) is already solved in the nonrelativistic case by the semigroup solutions (21) of the dynamical equation with the boundary condition (13). It is also the special case (22) of the transformation property (15). However the problem (ii) of probabilities "faster than light" can only be fully discussed in a relativistic theory using Poincaré transformations and not just by transformations generated by the Hamiltonian.

Specializing (16) to space–time translations $(1, x) = (1, (t, \mathbf{x}))$, we obtain from the space–time translated probability amplitude in analogy to (22):

$$\begin{aligned} |\langle\psi^-(x)|\hat{\mathbf{p}}, \xi(\mathbf{s}_R, j)^-\rangle &= \langle U(1, x)\psi^-|\hat{\mathbf{p}}, \xi(\mathbf{s}_R, j)^-\rangle \\ &= e^{-i\gamma(M_R - i\Gamma/2)(t - \mathbf{x}\cdot\mathbf{v})}\langle\psi^-|\hat{\mathbf{p}}, \xi(\mathbf{s}_R, j)^-\rangle \quad \text{for all} \quad \psi^- \in \Phi+ \end{aligned} \tag{25}$$

but only for $t \geq 0$ and $t^2 \geq \mathbf{x}^2 = r^2/c^2$. Therefore the probability for a decay event of the space–time translated decaying state $U_+^\times(1, (t, \mathbf{x}))\psi^G$,

$$|\langle\psi^-(x)|\hat{\mathbf{p}}, \xi(\mathbf{s}_R, j)^-\rangle|^2 = e^{-i\gamma(t-\mathbf{x}\cdot\mathbf{v})\Gamma}|\langle\psi^-|\hat{\mathbf{p}}, \xi(\mathbf{s}_R, j)^-\rangle|^2, \tag{26}$$

(and therefore detection of decay events) is only predicted to occur in the forward light cone (18)

$$t \geq 0 \quad \text{and} \quad t^2 \geq r^2/c^2 \text{ or } r/t < c. \tag{27}$$

This means the probability for decay events cannot propagate faster than the speed of light and Einstein causality is obeyed by Poincaré semigroup evolutions. Unitary representations of the Poincaré group in the Hilbert space or in the Schwartz space Φ, which fulfill (4), are not restricted to the forward light cone (18, 27) and therefore do not fulfill Einstein causality.

The Hardy space hypothesis allows predictions of probabilities only for Poincaré semigroup evolution in the forward light cone and as a consequence both the causality conditions "no registration of an observable in a state before that state has been prepared" and the Einstein causality condition "no propagation of probabilities with a speed faster than light" are fulfilled by probabilities for transitions between states and observables, and in particular for the relativistic Gamow vector.

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